Due Fri

Set w/2 ops 10 axioms — vector space w/ defined inner product (4 axioms)

6.2 – Angle and Orthogonality in Inner Product Spaces

Definition: The angle $\boldsymbol{\theta}$ **between vectors u and v** in a real inner product space V is $\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$. (As we saw with the dot product)

#5 Find the cosine of the angle between A and B with respect to the standard inner product on M_{22}.

$A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

$\langle A, B\rangle = tr(A^T B) = 2(3)+6(2)+1(1)-3(0) = 19$

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{2^2+6^2+1^2+(-3)^2} = \sqrt{50} = 5\sqrt{2}$

$\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{3^2+2^2+1^2+0^2} = \sqrt{14}$

$\cos\theta = \frac{19}{10\sqrt{7}} \approx 0.72$ $(\theta \simeq 44°)$

Theorem 6.2.1 Cauchy-Schwarz Inequality (generalization of Theorem 3.2.4)
If **u** and **v** are vectors in a real inner product space V, then $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

$\Rightarrow -1 \le \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|} \le 1$

Again a necessary result in order to consider the angle between two vectors in an inner product space.

Definition: Two vectors **u** and **v** in a real inner product space V are **orthogonal** if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

#10 Show that the vectors are orthogonal with respect to the standard inner product on P_2.
$\mathbf{p} = 2 - 3x + x^2, \mathbf{q} = 4 + 2x - 2x^2$

Clearly $\vec{p} \neq \vec{0}$, $\vec{q} \neq \vec{0}$.

$\langle \vec{p}, \vec{q}\rangle = 2(4) - 3(2) + 1(-2) = 0$

Theorem 6.2.2 Triangle inequalities (generalization of Theorem 3.2.5)

If **u**, **v**, and **w** are vectors in a real inner product space V, then:

a. $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)

b. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangle inequality for distances)

pf a) $\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle$

$= \langle \vec{u}, \vec{u} \rangle + \underline{2\langle \vec{u}, \vec{v} \rangle} + \langle \vec{v}, \vec{v} \rangle$

$\leq \langle \vec{u}, \vec{u} \rangle + \underline{2|\langle \vec{u}, \vec{v} \rangle|} + \langle \vec{v}, \vec{v} \rangle$ prop. of abs. value

$\leq \|\vec{u}\|^2 + \underline{2\|\vec{u}\|\|\vec{v}\|} + \|\vec{v}\|^2$ Cauchy-Schwarz inequality

$(\|\vec{u}\| + \|\vec{v}\|)^2$

Taking square roots finishes the proof.

b) $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \|(\vec{u}-\vec{w}) + (\vec{w}-\vec{v})\|$

$\leq \|\vec{u}-\vec{w}\| + \|\vec{w}-\vec{v}\|$

$= d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$

#33 Let $C[-1, 1]$ have the integral inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$ and let $\mathbf{p} = p(x) = x^2 - x$ and $\mathbf{q} = q(x) = x + 1$.

a. Find $\langle \mathbf{p}, \mathbf{q} \rangle$.

b. Find $\|\mathbf{p}\|$ and $\|\mathbf{q}\|$.

a. $\langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} (x^3+x^2-x^2-x)\,dx = \int_{-1}^{1} (x^3-x)\,dx = 0$

$\vec{p} \perp \vec{q}$ (odd function and symmetric interval)

b. $\|\vec{p}\| = \sqrt{\langle \vec{p}, \vec{p} \rangle} = \left[2\int_{\cancel{-1}\,0}^{1} (x^4 - \cancel{2x^3} + x^2)\,dx \right]^{1/2} = 2\left(\frac{1}{5} + \frac{1}{3}\right)$

$= \sqrt{\frac{16}{15}} = \frac{4}{\sqrt{15}}$

c. $\|\vec{q}\| = \left[2\int_{\cancel{-1}\,0}^{1} (x^2 + \cancel{2x} + 1)\,dx \right]^{1/2} = 2\left(\frac{1}{3} + 1\right)$

$= \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$

Explore: What is $\|\mathbf{p} + \mathbf{q}\|^2$? What about $\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$?

$\vec{p} + \vec{q} = x^2 + 1 \Rightarrow \|\vec{p} + \vec{q}\|^2 = \int_{-1}^{1} (x^2+1)\,dx$

$\langle \vec{p}+\vec{q}, \vec{p}+\vec{q} \rangle$

$= 2\int_{\cancel{-1}\,0}^{1} (x^4 + 2x^2 + 1)\,dx$

$= 2\left(\frac{1}{5} + \frac{2}{3} + 1\right) = \frac{56}{15}$

$\|\vec{p}\|^2 + \|\vec{q}\|^2 = \frac{16}{15} + \frac{8}{3} = \frac{56}{15}$

Here we have $\|\vec{p} + \vec{q}\|^2 = \|\vec{p}\|^2 + \|\vec{q}\|^2$

Theorem 6.2.3 Generalized Theorem of Pythagoras

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in a real inner product space, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Pf: $\|\vec{u} + \vec{v}\|^2 = \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle$

$= \|\vec{u}\|^2 + 2\cancel{\langle \vec{u}, \vec{v} \rangle} + \|\vec{v}\|^2$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2$ ✓

Example: Let $\mathbf{p} = p(x) = 1 - 2x^2$, $\mathbf{q} = q(x) = 4 - 2x + x^2$, $\mathbf{r} = r(x) = x + 2x^2$ and let P_2 have the standard inner product.

a. Compute the following: $\langle \mathbf{p},\ \mathbf{q}\rangle$ and $\langle \mathbf{q},\ \mathbf{r}\rangle$.

b. Now let P_2 have the inner product $\langle \mathbf{p},\ \mathbf{q}\rangle = a_0b_0 + a_1b_1 + ka_2b_2$. Find k so that $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.

c. Using this new inner product, compute $\langle \mathbf{q},\ \mathbf{r}\rangle$.

a. $\langle \vec{p}, \vec{q}\rangle = 1(4) - 2(1) = 2$

$\langle \vec{q}, \vec{r}\rangle = -2(1) + 1(2) = 0$ $\quad \vec{q} \perp \vec{r}$

b. $\langle \vec{p}, \vec{q}\rangle = 1(4) + k(-2)(1) = 4 - 2k$

$\langle \vec{p}, \vec{q}\rangle = 0 \Rightarrow k = 2$ $\quad \vec{p} \perp \vec{q}$

c. $\langle \vec{q}, \vec{r}\rangle = -2(1) + 2(1)(2) = 2$ $\quad \vec{q} \not\perp \vec{r}$

(a) & (b) describe 2 different inner product spaces.

Orthogonality depends on the vectors and the inner product being used.

#17 Do there exist scalars k and l such that the vectors $\mathbf{p}_1 = 2 + kx + 6x^2$, $\mathbf{p}_2 = l + 5x + 3x^2$, $\mathbf{p}_3 = 1 + 2x + 3x^2$ are mutually orthogonal with respect to the standard inner product on P_2?

all pairs are orthogonal

$\langle \vec{p}_1, \vec{p}_2\rangle = 2l + 5k + 18$ If $\vec{p}_1 \perp \vec{p}_2$, $2l + 5k + 18 = 0$

$\langle \vec{p}_1, \vec{p}_3\rangle = 2 + 2k + 18$. If $\vec{p}_1 \perp \vec{p}_3$, $20 + 2k = 0 \Rightarrow k = -10$

$\langle \vec{p}_2, \vec{p}_3\rangle = l + 10 + 9$. If $\vec{p}_2 \perp \vec{p}_3$, $l = -19$.

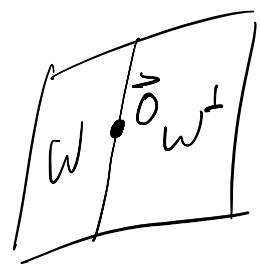
But if $k = -10$, $l = -19$, then

$\langle \vec{p}_1, \vec{p}_2\rangle = -38 - 50 + 18 \neq 0$

No k & l work.

Definition: (inner product space analog of Definition 2 in Section 4.9) If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^{\perp}$ (pronounced "W perp").

Theorem 6.2.4 (generalization of Theorem 4.9.6 (a) and (b))
If W is a subspace of a real inner product space V, then:
a) $W^{\perp}$ is a subspace of V.
b) $W \cap W^{\perp} = \{\mathbf{0}\}$.



Theorem 6.2.5 (generalization of Theorem 4.9.6(c))
If W is a subspace of a real finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W, that is, $(W^{\perp})^{\perp} = W$.

Euclidean inner product
a.k.a. dot product

#27 Find a basis for the orthogonal complement of the subspace of R^4 spanned by the vectors $\mathbf{v}_1 = (1,\ 4,\ 5,\ 2)$, $\mathbf{v}_2 = (2,\ 1,\ 3,\ 0)$, $\mathbf{v}_3 = (-1,\ 3,\ 2,\ 2)$.

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -2/7 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_3 + \frac{2}{7}x_4 \quad , \quad x_2 = -x_3 - \frac{4}{7}x_4$

Let $s = x_3$ and $t = x_4$

$$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix} t$$

a basis for the orthogonal complement is

$\{(1,1,-1,0), (2,-4,0,7)\}$.

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solutions are in $\mathbb{R}^3$

$$A\vec{x} = \vec{0}$$

But the sol'n space to a homog. system is the orthogonal complement to the row space of A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0$$

Pf of Cauchy-Schwarz inequality

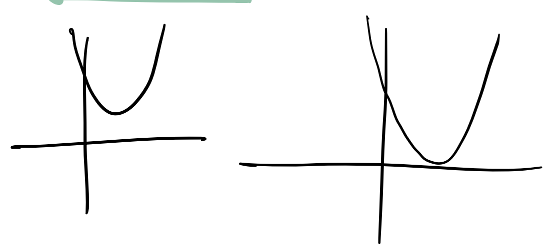
That is, pf that for $\vec{u}, \vec{v} \in V$ an inner product space, $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$.

If $\vec{u} = \vec{0}$, then $\langle \vec{u}, \vec{v} \rangle = 0$ and $\|\vec{u}\| = 0$.

For $\vec{u} \neq \vec{0}$, let $\langle \vec{u}, \vec{u} \rangle = a$, $2\langle \vec{u}, \vec{v} \rangle = b$, $\langle \vec{v}, \vec{v} \rangle = c$, and t be any real number.

$$0 \leq \langle t\vec{u} + \vec{v}, t\vec{u} + \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle t^2 + 2\langle \vec{u}, \vec{v} \rangle t + \langle \vec{v}, \vec{v} \rangle$$

$$\Rightarrow at^2 + bt + c \geq 0$$



Thus $b^2 - 4ac \leq 0$

$$\Rightarrow 4\langle \vec{u}, \vec{v} \rangle^2 - 4\langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle \leq 0$$

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle$$ one form

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$$ another form

$$\Rightarrow |\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\| \quad \checkmark$$ yet another form